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Q No → Show that the function $f(z) = u + iv$,

where, $f(z) = \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$; $(z \neq 0)$, $f(0) = 0$.

is Continuous and that the Cauchy-Riemann equations are satisfied at the origin, yet $f'(0)$ does not exist.

Solun. We have

$$f(z) = u + iv = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$$

$$= \frac{x^3 - y^3}{x^2+y^2} + i\frac{(x^3+y^3)}{(x^2+y^2)}$$

$$\therefore u = \frac{x^3 - y^3}{x^2+y^2} \quad \& \quad v = \frac{x^3+y^3}{x^2+y^2}.$$

Since, u & v are rational function of x & y with non-zero denominators,

Therefore, u & v are Continuous at $z \neq 0$,

Hence, $f(z)$ is Continuous at $z \neq 0$

Now, we test the Continuity of $f(z)$ at $z=0$. Changing to Polar Co-ordinate, we have $u = r(\cos^3\theta - \sin^3\theta)$ and $v = r(\cos^3\theta + \sin^3\theta)$.

Clearly, $u \to 0$, $v \to 0$ then $r \to 0$ what every value of θ may be, Since $f(0)=0$ the actual value of u & v at the origin will be $(0,0)$, therefore u & v, then $f(z)$ are Continuous at $z=0$.

Now, $$\frac{\partial u}{\partial x} = \underset{x\to 0}{Lt}\ \frac{u(x,0)-u(0,0)}{x} = \underset{x\to 0}{Lt}\ \frac{\frac{x^3}{x^2}-0}{x} = 1$$

$$\frac{\partial u}{\partial y} = \underset{y\to 0}{Lt}\ \frac{u(0,y)-u(0,0)}{y} = \underset{y\to 0}{Lt}\ \frac{-\frac{y^3}{y^2}-0}{y} = -1.$$

$$\frac{\partial v}{\partial y} = \underset{y\to 0}{Lt}\ \frac{v(0,y)-v(0,0)}{y} = \underset{y\to 0}{Lt}\ \frac{\frac{y^3}{y^2}-0}{y} = 1.$$

$$\frac{\partial v}{\partial x} = \underset{x\to 0}{Lt}\ \frac{v(x,0)-v(0,0)}{x} = \underset{x\to 0}{Lt}\ \frac{\frac{x^3}{x^2}-0}{x} = 1.$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

Therefore, Cauchy Riemann equn. are Satisfied

$$f'(0) = \underset{z\to 0}{Lt}\ \frac{f(z)-f(0)}{z} = \underset{\substack{x\to 0\\ y\to 0}}{Lt}\ \frac{\frac{x^3-y^3}{x^2+y^2} + i\,\frac{x^3+y^3}{x^2+y^2}}{x+iy}$$

$$= \underset{\substack{x\to 0\\ y\to 0}}{Lt}\ \frac{(x^3-y^3)+i(x^3+y^3)}{(x^2+y^2)(x+iy)}$$

When $z \to 0$ along the line $y = x$ the above limit becomes.

$$\cancel{f'(0) = \underset{x\to 0}{Lt}\ x^3}\ \underset{x\to 0}{Lt}\ \frac{x^3-x^3+i(x^3+x^3)}{(x^2+x^2)(x+ix)} = \underset{x\to 0}{Lt}\ \frac{2ix^3}{2(1+i)x^3}$$

$$= \frac{(1-i)}{2}.$$

When, $z \to 0$ along x-axis i.e. $y = 0$ the above limit becomes,

$$f'(0) = \underset{x \to 0}{Lt} \frac{x^3 + i x^3}{x^3} = (1+i).$$

Hence, $f'(0)$ is not unique therefore $f'(z)$ is not analytic at $z = 0$.

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